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APPROXIMATIONS IN NUMERICAL ANALYSIS
A REPORT ON A STUDY

Cecil Hastings, Jr.

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The RAND Corporation
1700 MAIN ST. · SANTA MONICA · CALIFORNIA

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Approximations in Numerical Analysis

A Report on a Study

Dr. D. H. Lehmer has asked us, on behalf of the Committee on Arrangements, to report to the American Mathematical Society on our study concerning the practical approximation of functions. It is with some embarrassment that we undertake to comply with this request. Ours is not a theoretical research study, but rather a practical one of a developmental nature in which only very elementary mathematics has been employed. Our principal tools have been the laborious use of the desk calculator, and the occasional use of a little imagination.

<u>The Nature and Purpose of the Study</u>:

The introduction of the high speed digital computing machine makes it desirable for the numerical analyst to acquire a fresh outlook on the problem of what might be called the numerical representation of a function. In olden days, when the principal instrument of calculation was the pencil attached to a human being, the conventional table equipped with differences or the graph that could be read by eye were reasonable means of representing a function numerically. The high speed digital computing machine, however, has quite different needs from those of the human computer. In the first place, the digital computing machine can't read books, charts, and graphs. All of the numerical information it has access to must either be stored in its memory, which may be quite limited, or be fed into its memory as required. On the other hand, the digital computing machine can do vast quantities of simple arithmetic very quickly and efficiently.

For these reasons, an obvious means of functional representation for the machine is that of using compact analytical approximations that can be evaluated easily in terms of basic machine operations.

It was quite apparent that a great deal of consideration would be given to the subject of polynomial approximation, and this has indeed turned out to be the case. But the numerical analyst will be severely handicapped if he limits himself to the use of polynomial forms alone. Rational forms are at times a must, if a reasonably simple representation is to be obtained. He will be further handicapped if he does not incorporate an occasional square root or logarithm in his parametric form when such is definitely a requirement.

Thus, the most important objective of our study was that of demonstrating how to select the appropriate type of parametric form for use in a given instance of approximation. There seemed to be only one means of accomplishing this desired end. We would have to work out a large number of illustrative examples, and this is what we have been doing for a number of years now. Some of the resulting approximations are fortunately quite useful, and a number of these have seen considerable use in many computing installations.

We are then, concerned with the problem of choice of parametric form, with the problem of determining the coefficients in the parametric form, with the problem of presenting our results in such a form that they will be readily understood, and finally with the problem of communicating the experience that we have gained in the preparation of our examples to the people who could

profit by this experience.

Choice of Parametric Form:

As stated above, the problem of choice of parametric form is one that can only effectively be discussed in terms of concrete examples. For this reason, we now discuss half a dozen such examples. As we are only concerned with the form of the approximation at present, the laboriously obtained numerical constants have been replaced by letters.

* * *

Example (1) The Common Logarithmic Function:

To better than .000035 over $1 \leq x \leq 10$,

$$\lg x \doteq \tfrac{1}{2} + c_1\eta + c_3\eta^3 + c_5\eta^5 , \quad \eta = \frac{x - \sqrt{10}}{x + \sqrt{10}} .$$

The choice of form here was governed by two considerations. In the first place, it seemed apparent that a simple polynomial form could not do the job very efficiently, and hence a rational form was indicated. Secondly, individual terms of the selected rational form were chosen to have the property $\psi(10/x) = -\psi(x)$ so that the form would mirror an important property of the logarithmic function. As a consequence of this later consideration, the parameters c_i could be determined solely by approximation over the reduced interval $(1, \sqrt{10})$.

Example (2) The Gaussian Error Integral:

To better than .00002 over $0 \leq x \leq \infty$,

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \doteq 1 - \left[1 + a_1x + \ldots + a_5x^5\right]^{-8} .$$

A polynomial to a substantial negative power has been found

efficient for use in the approximation of functions over $(0, \infty)$, in which the asymptotic behavior is roughly that of a decaying exponential.

Example (3) A Bessel Function:

To better than .00005 over $0 \leq x \leq \infty$,

$$e^{-x} I_o(x) \doteq \sqrt{\frac{1 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}}.$$

The choice of form here was governed by the asymptotic behavior, $e^{-x} I_o(x) \sim (2\pi x)^{-\frac{1}{2}}$, of the function as $x \rightarrow \infty$.

Example (4) A Certain Definite Integral:

To better than .00015 over $0 \leq z \leq \infty$,

$$W(z) = \int_0^{\infty} \frac{e^{-uz}}{K_1^2(u) + \pi^2 I_1^2(u)} \frac{du}{u} \doteq \frac{1+a_1 z}{2+b_1 z+b_2 z^2+b_3 z^3}.$$

The choice of form here was governed by the asymptotic behavior $W(z) \sim z^{-2}$ as $z \rightarrow \infty$. This integral appears in a paper by G. N. Ward entitled 'The Approximate External and Internal Flow Past a Quasi-Cylindrical Tube Moving at Supersonic Speeds.' A brief table of W(z) is contained in this paper. We take this opportunity to thank Dr. E. T. Goodwin of the National Physical Laboratory, Teddington, Middlesex and his associates for re-tabulating W(z) to 9D over $(0, \infty)$ for our use in this project.

Example (5) The Offset Circle Probability Function q(R, x):

To better than .00014 over $-\infty \leq x \leq \infty$,

$$q(1, x) = \int_1^{\infty} e^{-\frac{1}{2}(\rho^2 + x^2)} I_0(\rho x) \rho d\rho \doteq 1 - a(1+b_2 x^2+b_4 x^4+b_6 x^6)^{-4}.$$

The choice of form here was governed by the fact that the function approximated is an even function of x and by the further

consideration already stated in Example (2). The $q(R,x)$ function has been finely tabulated to 6D by the joint effort of NBS INA and RAND.

Example (6) A Scoring Camera Data Reduction Formula:

To about as good as the data in the case studied,

$$d(r) = \frac{\tan\theta}{r} \doteq \frac{a_0 + a_2r^2 + a_4r^4}{1 - b_2r^2} .$$

Here $r = r(\theta)$ is an empirically determined function relating the angle θ at which a ray enters the camera to the measured distance r on the film of the image from the origin determihed by the intersection of the optical axis of the lens system with the surface of the film. The choice of the approximating form was governed by the fact that $d(r)$ must mathematically be considered an even function of r, and by the further fact that the singularity introduced by $\tan\theta$ should be properly mirrored in the chosen form. The problem was submitted to us by Mr. John Lowe of the Douglas Aircraft Company.

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While the above examples may give some interesting clues to the reader, they may also give an unfortunate impression that the problem of choice of form is a simpler affair than it actually is. It takes quite a bit of experience in the approximation of functions before one gets a feeling for just what aspects of the function to be approximated should be mirrored in the chosen form. Even given the required experience, there will be times when a form appears to be chosen properly, and yet is incapable of doing the job required of it. But this is all a part of the

game, and should not deter the practical numerical analyst from thinking more constructively about the problem of choice of form.

<u>The Form (15) Sheets</u>:

In order to make certain concrete results of our study available, we began in the early days of our study, to prepare individual sheets [1] each containing data pertinent to an interesting approximation and including a carefully drawn error curve. These are the so-called Form (15) sheets. To date, some seventy such sheets have been prepared, and some five hundred binders of these sheets have been distributed to members of the computing fraternity. Only one formal notice [2] concerning the existence of this binder has been made, and this was made quite recently.

In addition to preparation of the Form (15) sheets, we have also initiated a cumulative publication of approximations in the journal MTAC, with Dr. Lehmer's blessing. At this writing, forty five new approximations have been submitted to MTAC for publication, and twelve of these items [3] have already appeared in print.

<u>Reports on Methods</u>:

To date we have prepared very little written material about methods of approximation in a form suitable for distribution. Quite naturally, a certain amount of elementary technique has been developed for use in carrying out the fitting of the parametric forms to the functions in question. These techniques are of an entirely numerical nature. By this we mean that, as a first step, we require a numerical table of the function to be approximated.

Then, we determine the numerical values of the parameters in the parametric form by a process of trial and error that involves the numerical solution of equations, graphing of error curves, etc. One preliminary paper [4], written while our study was in a very early stage, gives some idea of our methods. A recent effort [5] of ours gives a little further information. We shall devote a few words to this latter report. Reference [5] has the rather flippant title 'The Incomplete Approximator' (In six fits). This little effort started its life as a film strip. We prepared a sequence of one hundred thirteen large inked drawings, and had these photographed in sequence on a continuous strip of movie film. This film strip was presented to the June 12, 1953 meeting of the Digital Computers Association. Later, the original drawings were revised, photographically reduced in size and the resulting photos mounted on sheets of paper together with a type-written running commentary. Thus the finished report resembled a comic book in appearance, and we were quite pleased with the admittedly crude 'cartoon strip' technique. A limited number of copies of this report were offset printed, and distributed to members of the Digital Computers Association.

Preparation of an Omnibus Report:

At present we are working on the preparation of an omnibus report which will serve to present our thoughts to date on the subject of practical approximation. The material contained in our binder of approximations will be included as an appendix to thisreport.

It is our present plan to prepare the text of this report

in cartoon strip style. The result will be somewhat of a novelty but should serve to put across what we have to say in an enjoyable and palatable manner. At present, we are working on the development of the cartoon strip technique as applied to our particular needs. We are trying to learn to convey ideas of importance in as efficient a manner as possible. Towards this end, we have been giving some consideration to the idea of trying to describe the actual thinking process that should take place in the reader's mind. We hope in this fashion, to transfer some of the 'frame of mind' that a person should have in attacking a practical problem in approximation.

The RAND Corporation
1700 Main Street
Santa Monica, California

Mr. Cecil Hastings, Jr.
Mrs. David K. Hayward
Mr. James P. Wong, Jr.

References Resulting from the Study

[1] The RAND Corporation Approximations in Numerical Analysis (A binder of some seventy loose sheets.)

[2] The RAND Collection of Illustrative Approximations MTAC v.6, Note 139, p.251-253.

[3] Analytical Approximations MTAC v.7, Note 143, p. 67-69.

[4] Rational Approximation in High-Speed Computing, Proceedings Computation Seminar December 1949, International Business Machines Corporation, p. 57-61.

[5] The Incomplete Approximator (In six fits) (A Paper presented to the June 12, 1953 meeting of the Digital Computers Association.)